

Scattering and correlations

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Introduction

Let us consider the propagation of scalar waves with the speed $v > 0$ given by the wave equation $u_{tt} - v^2 \Delta u = 0$ outside a compact domain D in the Euclidean space \mathbb{R}^d . Let us put $\Omega = \mathbb{R}^d \setminus D$. We can assume for example Neumann boundary conditions. We will denote by Δ_Ω the (self-adjoint) Laplace operator with the boundary conditions. So our stationary wave equation is the Helmholtz equation

$$v^2 \Delta_\Omega f + \omega^2 f = 0 \quad (1)$$

with the boundary conditions. We consider a bounded interval $I = [\omega_-^2, \omega_+^2] \subset]0, +\infty[$ and the Hilbert subspace \mathcal{H}_I of $L^2(\Omega)$ which is the image of the spectral projector P_I of our operator $-v^2 \Delta_\Omega$.

Let us compute the integral kernel $\Pi_I(x, y)$ of P_I defined by:

$$P_I f(x) = \int_\Omega \Pi_I(x, y) f(y) |dy|$$

into 2 different ways:

1. From general spectral theory
2. From scattering theory.

Taking the derivatives of $\Pi_I(x, y)$ w.r. to ω_+ , we get a simple general and exact relation between the correlation of scattered waves and the Green's function confirming the calculations from [2] in the case where D is a disk. See also [1] for other relations between correlations and Green's functions used in passive imaging.

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1 $\Pi_I(x, y)$ from spectral theory

Using the resolvent kernel (Green's function) $G(\omega, x, y) = [(\omega^2 + v^2 \Delta_\Omega)^{-1}](x, y)$ for $\text{Im} \omega > 0$ and the Stone formula, we have:

$$\Pi_I(x, y) = -\frac{2}{\pi} \text{Im} \left(\int_{\omega_-}^{\omega_+} G(\omega + i0, x, y) \omega d\omega \right)$$

Taking the derivative w.r. to ω_+ of $\Pi_{[\omega_-^2, \omega_+^2]}(x, y)$, we get

$$\frac{d}{d\omega} \Pi_{[\omega_-, \omega_+^2]}(x, y) = -\frac{2\omega}{\pi} \text{Im}(G(\omega + i0, x, y)) . \quad (2)$$

2 Short review of scattering theory

There are many references for scattering theory: for example [3, 4].

Let us define for $\mathbf{k} \in \mathbb{R}^d$ the plane wave

$$e_0(x, \mathbf{k}) = e^{i\langle \mathbf{k} | x \rangle} .$$

We are looking for solutions

$$e(x, \mathbf{k}) = e_0(x, \mathbf{k}) + e^s(x, \mathbf{k})$$

of the Helmholtz equation (1) in Ω where e^s , the scattered wave satisfies the so-called Sommerfeld radiation condition¹:

$$e^s(x, \mathbf{k}) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} \left(e^\infty\left(\frac{x}{|x|}, \mathbf{k}\right) + O\left(\frac{1}{|x|}\right) \right), \quad x \rightarrow \infty .$$

The complex function $e^\infty(\hat{x}, \mathbf{k})$ is usually called the *scattering amplitude*.

It is known that the previous problem admits a unique solution. In more physical terms, $e(x, \mathbf{k})$ is the wave generated by the full scattering process from the plane wave $e_0(x, \mathbf{k})$. Moreover we have a generalized Fourier transform:

$$f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{f}(\mathbf{k}) e(x, \mathbf{k}) |d\mathbf{k}|$$

with

$$\hat{f}(\mathbf{k}) = \int_{\mathbb{R}^d} \overline{e(y, \mathbf{k})} f(y) |dy| .$$

From the previous generalized Fourier transform, we can get the kernel of any function $\Phi(-v^2 \Delta_\Omega)$ as follows:

$$[\Phi(-v^2 \Delta_\Omega)](x, y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \Phi(v^2 k^2) e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\mathbf{k}| . \quad (3)$$

¹As often, we denote $k := |\mathbf{k}|$ and $\hat{\mathbf{k}} := \mathbf{k}/k$

3 $\Pi_I(x, y)$ from scattering theory

Using Equation (3) with $\Phi = 1_I$ the characteristic functions of some bounded interval $I = [\omega_-^2, \omega^2]$, we get:

$$\Pi_I(x, y) = (2\pi)^{-d} \int_{\omega_- \leq vk \leq \omega} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\mathbf{k}| .$$

Using polar coordinates and defining $|d\sigma|$ as the usual measure on the unit $(d-1)$ -dimensional sphere, we get:

$$\Pi_I(x, y) = (2\pi)^{-d} \int_{\omega_- \leq vk \leq \omega} k^{d-1} dk \int_{\mathbf{k}^2 = k^2} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\sigma| .$$

We will denote by σ_{d-1} the total volume of the unit sphere in \mathbb{R}^d : $\sigma_0 = 2$, $\sigma_1 = 2\pi$, $\sigma_2 = 4\pi, \dots$.

Taking the same derivative as before, we get:

$$\frac{d}{d\omega} \Pi_{[\omega_-^2, \omega^2]}(x, y) = (2\pi)^{-d} \frac{\omega^{d-1}}{v^d} \int_{vk=\omega} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\sigma| .$$

Let us look at $e(x, \mathbf{k})$ as a random wave with $k = \omega/v$ fixed. The point-point correlation of such a random wave $C_\omega^{\text{scatt}}(x, y)$ is given by:

$$C_\omega^{\text{scatt}}(x, y) = \frac{1}{\sigma_{d-1}} \int_{vk=\omega} e(x, \mathbf{k}) \overline{e(y, \mathbf{k})} |d\sigma| .$$

Then we have:

$$\frac{d}{d\omega} \Pi_{[\omega_-^2, \omega^2]}(x, y) = (2\pi)^{-d} \frac{\omega^{d-1} \sigma_{d-1}}{v^d} C_\omega^{\text{scatt}}(x, y) . \quad (4)$$

4 Correlation of scattered plane waves and Green's function: the scalar case

From Equations (2) and (4), we get:

$$(2\pi)^{-d} \frac{\omega^{d-1} \sigma_{d-1}}{v^d} C_\omega^{\text{scatt}}(x, y) = -\frac{2\omega}{\pi} \text{Im}(G(\omega + i0, x, y)) .$$

Hence, we have

Theorem 1 *For the scalar wave equation $u_{tt} - v^2 \Delta u = 0$ outside a bounded domain in \mathbb{R}^d , we have the following expression of the correlation of scattered wave of frequency ω in terms of the Green's function:*

$$C_\omega^{\text{scatt}}(x, y) = -\frac{2^{d+1} \pi^{d-1} v^d}{\sigma_{d-1} \omega^{d-2}} \text{Im}(G(\omega + i0, x, y)) .$$

For later use, we put

$$\gamma_d = \frac{2^{d+1} \pi^{d-1}}{\sigma_{d-1}} . \quad (5)$$

5 The case of elastic waves

We will consider the elastic wave equation in the domain Ω :

$$\hat{H}\mathbf{u} - \omega^2\mathbf{u} = 0,$$

with self-adjoint boundary conditions. We will assume that, *at large distances*, we have

$$\hat{H}\mathbf{u} = -a \Delta\mathbf{u} - b \operatorname{grad} \operatorname{div} \mathbf{u}.$$

where a and b are constants:

$$a = \frac{\mu}{\rho}, \quad b = \frac{\lambda + \mu}{\rho}$$

with λ, μ the Lamé's coefficients and ρ the density of the medium. We will denote $v_P := \sqrt{a+b}$ (resp. $v_S := \sqrt{a}$) the speeds of the P –(resp. S –)waves near infinity.

5.1 The case $\Omega = \mathbb{R}^d$

We want to derive the spectral decomposition of \hat{H} from the Fourier inversion formula. Let us choose, for $\mathbf{k} \neq 0$, by $\hat{\mathbf{k}}, \hat{\mathbf{k}}_1, \dots, \hat{\mathbf{k}}_{d-1}$ an orthonormal basis of \mathbb{R}^d with $\hat{\mathbf{k}} = \frac{\mathbf{k}}{k}$ such that these vectors depends in a measurable way of \mathbf{k} . Let us introduce $P_P^{\mathbf{k}} = \hat{\mathbf{k}}\hat{\mathbf{k}}^\star$ the orthogonal projector onto $\hat{\mathbf{k}}$ and $P_S^{\mathbf{k}} = \sum_{j=1}^{d-1} \hat{\mathbf{k}}_j\hat{\mathbf{k}}_j^\star$ so that $P_P + P_S = \operatorname{Id}$. Those projectors correspond respectively to the polarizations of P – and S –waves.

We have

$$\begin{aligned} \Pi_I(x, y) = & (2\pi)^{-d} \int_{\omega^2 \in I} \omega^{d-1} d\omega \left(v_P^{-d} \int_{v_P k = \omega} e^{i\mathbf{k}(x-y)} P_P^{\mathbf{k}} d\sigma + \right. \\ & \left. v_S^{-d} \int_{v_S k = \omega} e^{i\mathbf{k}(x-y)} P_S^{\mathbf{k}} d\sigma \right). \end{aligned}$$

using the plane waves

$$e_P^O(x, \mathbf{k}) = e^{i\mathbf{k}x} \hat{\mathbf{k}}$$

and

$$e_{S,j}^O(x, \mathbf{k}) = e^{i\mathbf{k}x} \hat{\mathbf{k}}_j$$

we get the formula²:

$$\begin{aligned} \Pi_I(x, y) = & (2\pi)^{-d} \int_{\omega^2 \in I} \omega^{d-1} d\omega \left(v_P^{-d} \int_{v_P k = \omega} |e_P^O(x, \mathbf{k})\rangle \langle e_P^O(y, \mathbf{k})| d\sigma + \right. \\ & \left. v_S^{-d} \sum_{j=1}^{d-1} \int_{v_S k = \omega} |e_{S,j}^O(x, \mathbf{k})\rangle \langle e_{S,j}^O(y, \mathbf{k})| d\sigma \right). \end{aligned}$$

²We use the “bra-ket” notation of quantum mechanics: $|e\rangle\langle f|$ is the operator $x \rightarrow \langle f|x\rangle e$ where the brackets are linear w.r. to the second entry and anti-linear w.r. to the first one

5.2 Scattered plane waves

There exists scattered plane waves

$$\begin{aligned} e_P(x, \mathbf{k}) &= e_P^O(x, \mathbf{k}) + e_P^s(x, \mathbf{k}) \\ e_{S,j}(x, \mathbf{k}) &= e_{S,j}^O(x, \mathbf{k}) + e_{S,j}^s(x, \mathbf{k}) \end{aligned}$$

satisfying the Sommerfeld condition and from which we can deduce the spectral decomposition of \hat{H} .

5.3 Correlations of scattered plane waves and Green's function

Following the same path as for scalar waves, we get an identity which holds now for the full Green's tensor $\text{Im}\mathbf{G}(\omega + iO, x, y)$:

Theorem 2 *For the elastic wave equation, we have the following expression of the imaginary part of the Green's function in terms of the correlation of the scattered S- and P- waves:*

$$\begin{aligned} \text{Im}\mathbf{G}(\omega + iO, x, y) &= -\gamma_d^{-1}\omega^{d-2} \left(\frac{1}{\sigma_{d-1}v_P^d} \int_{v_P\mathbf{k}=\omega} |e_P(x, \mathbf{k})\rangle \langle e_P(y, \mathbf{k})| d\sigma + \right. \\ &\quad \left. \frac{1}{\sigma_{d-1}v_S^d} \sum_{j=1}^{d-1} \int_{v_S\mathbf{k}=\omega} |e_{S,j}(x, \mathbf{k})\rangle \langle e_{S,j}(y, \mathbf{k})| d\sigma \right) , \end{aligned}$$

with γ_d defined by Equation (5).

This formula expresses the fact that the correlation of scattered plane waves randomized with the appropriate weights (v_P^{-d} versus v_S^{-d}) is proportional to the Green's tensor. Let us insist on the fact that this is true everywhere in Ω even in the domain where a and b are not constants.

References

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